Appendix 2.1

1. **Drawing a continuous function.** From a geometric point of view, a function is continuous on an interval if you can "draw it's graph without lifting your pencil from the paper". For, if you have to lift your pencil from the graph there has to be a "jump" in the graph. At that point the function is not continuous (the limits of the function as you approach the jump from the right and from the left, if they exist, will be different). But be careful, any curve you can draw will be 'smooth', except for a finite number of corners. Such a function will be differentiable except at the corners. Yet there exist continuous functions that are differentiable nowhere! They would be impossible to draw.

2. By previous results on Limits all the following functions are continuous on $\mathbb{R}$:

$$f_1(x) = \begin{cases} \frac{e^x-1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \qquad f_2(x) = \begin{cases} \frac{\sin\theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0. \end{cases}$$

$$f_3(x) = \begin{cases} \frac{\cos\theta-1}{\theta} & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0. \end{cases} \qquad f_4(x) = \begin{cases} \frac{\cos\theta-1}{\theta^2} & \text{if } \theta \neq 0 \\ -\frac{1}{2} & \text{if } \theta = 0. \end{cases}$$

$$f_5(x) = \begin{cases} \frac{e^x-1-x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0. \end{cases}$$

3. **Rational and irrational numbers in an interval**. In lectures we used the result that in any interval (a, b) we can find a rational number and we can also find an irrational number.

   Proof Let $\ell = b - a$ be the length of the interval. Choose $n \in \mathbb{N}$ so large that $2^{-n} < \ell$.

   Assume for the sake of a contradiction that for **no** $m \in \mathbb{Z}$ do we have $m/2^n \in (a, b)$. This means there exists $p \in \mathbb{N}$ for which

$$\frac{p}{2^n} \leq a < b \leq \frac{p+1}{2^n}.$$

   (In fact $p = \max\{m \in \mathbb{Z} : m \leq a2^n\}$.) Then, writing

$$\frac{p}{2^n} \leq a \text{ as } -a \leq -\frac{p}{2^n},$$

we find that

$$\ell = b - a \leq \frac{p+1}{2^n} - \frac{p}{2^n} = \frac{1}{2^n} < \ell.$$

The final result, a strict inequality $\ell < \ell$, is a contraction, and so our assumption is false, and so there **does** exist $m \in \mathbb{Z}$ for which the rational number $r = m/2^n$ lies in (a, b) for some $m \in \mathbb{N}$.

To find an irrational number do the above for the interval $\left(a/\sqrt{2},\, b/\sqrt{2}\right)$ to find a rational r_0 in this interval. You can then check that $r_0\sqrt{2}$ is an irrational number lying in (a, b). ■

4. **Example 2.1.16** *Show that the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) = \begin{cases} \sin\left(\dfrac{\pi}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

*is **not** continuous at* $x = 0$.

Solution Assume that g **is** continuous at $x = 0$. Then

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = g(0) = 0.$$

In particular this means the limit exists. Yet from a Problem Sheet we know that $\lim_{x\to 0} \sin(\pi/x)$ does **not** exist. This contradiction means our assumption is false and so f is not continuous at 0. ■

Note that there is, in fact, **no** value for $g(0)$ that would make the function continuous at $x = 0$.

5. **Example 2.1.17** *If*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

and $a \notin \mathbb{Q}$ *show that* f *is **not** continuous at* a.

Solution Assume f is continuous at a.

Choose $\varepsilon = 1/2$ in the definition of continuity to find $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < 1/2$.

But $a \notin \mathbb{Q}$ implies $f(a) = 0$ while in any interval, such as $(a, a + \delta)$ we can find a *rational* x_0 for which $f(x_0) = 1$. Thus

$$\frac{1}{2} > |f(x_0) - f(a)| = |1 - 0| = 1.$$

Contradiction, so our assumption is false. Thus f is not continuous at a. ■

6. **Product and Quotient Rules for continuous functions**. Assume that f and g are continuous at $a \in \mathbb{R}$. This means that $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. For the Product Rule

$$\begin{aligned}\lim_{x\to a} (fg)(x) &= \lim_{x\to a} (f(x)\,g(x)) \quad \text{from definition of } fg, \\ &= \lim_{x\to a} f(x) \lim_{x\to a} g(x) \quad \text{from Product Rule for } \textit{limits}, \\ &= f(a)\,g(a) = (fg)(a)\,.\end{aligned}$$

For the Quotient Rule, we have to also assume that $g(a) \neq 0$. Then

$$\begin{aligned}\lim_{x\to a} (f/g)(x) &= \lim_{x\to a} (f(x)/g(x)) \quad \text{from definition of } f/g, \\ &= \lim_{x\to a} f(x)/ \lim_{x\to a} g(x) \quad \text{from Quotient Rule for } \textit{limits}, \\ &= f(a)/g(a) = (f/g)(a)\,.\end{aligned}$$

In applying the Product and Quotient Rules for limits we should have observed that they were allowable since the individual limits, $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, and is non-zero in the case of the Quotient Rule.

7. **Interchanging a limit with other operations.** The Composite Rule for functions in the form

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

is an example of a whole class of results in Mathematical Analysis looking at when a limit can be taken inside a function, or an operation such as summation or integration.

For example under what conditions on the functions $f_n(x)$ can we say

$$\lim_{x\to a} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to a} f_n(x)?$$

The important point here is it is an infinite summation. By repeated application of the Sum Rule above we know this result holds for *finite* summations.

For a function of two variables under what conditions can we say

$$\lim_{x\to a}\int_{-\infty}^{\infty} f(x,t)\,dt = \int_{-\infty}^{\infty}\lim_{x\to a} f(x,t)\,dt?$$

Again for a function of two variables under what conditions can we interchange limits as in

$$\lim_{x\to a}\lim_{t\to b} f(x,t) = \lim_{t\to b}\lim_{x\to a} f(x,t)?$$

Unfortunately we cannot answer these questions in this course.

8. **Composition rule for limits.** I have missed out another Composition result, one for limits.

   Assume $\lim_{x\to a} g(x)$ exists, equal to L say. Assume not that f is continuous at L but only that $\lim_{y\to L} f(y)$ exists, equal to M say. What can be said of $\lim_{x\to a}(f\circ g)(x)$? Is it equal to M?

   There is a possible problem, for though the limit $\lim_{y\to L} f(y)$ exists, the value $f(L)$ may not. So in examining $(f\circ g)(x) = f(g(x))$ we would not want $g(x) = L$ for x close to a.

   Theorem 2.1.18 *Assume* $\lim_{x\to a} g(x) = L$ *exists and there exists a deleted neighbourhood of* a *on which* $g(x)\neq L$. *Assume* $\lim_{y\to L} f(y) = M$ *exists. Then* $\lim_{x\to a}(f\circ g)(x)$ *exists with value* M.

   Proof By assumption there exists $\delta_0 > 0$ such that if $0 < |x-a| < \delta_0$ then $g(x)\neq L$ or, in a form appropriate for us,

   $$0 < |g(x) - L|\,. \tag{6}$$

   Let $\varepsilon > 0$ be given. Look at f first to find $\delta_1 > 0$ such that

   $$0 < |y - L| < \delta_1 \implies |f(y) - M| < \varepsilon. \tag{7}$$

   Take $\varepsilon = \delta_1$ in the definition of $\lim_{x\to a} g(x) = L$ to find $\delta_2 > 0$ such that

   $$0 < |x-a| < \delta_2 \Longrightarrow |g(x) - L| < \delta_1. \tag{8}$$

   Choose $\delta = \min(\delta_0, \delta_2)$ and assume $0 < |x-a| < \delta$. Then we have both

- $0 < |g(x) - L|$ by (6) and
- $|g(x) - L| < \delta_1$ by (8).

Combine to get $0 < |g(x) - L| < \delta_1$. But then this implies $|f(g(x)) - M| < \varepsilon$ by (7) with $y = g(x)$. This verifies the definition of $\lim_{x\to a} (f \circ g)(x) = M$. ■

If it is not the case that there exists a deleted neighbourhood of a on which $g(x) \neq L$ then $f(L)$ has to be defined and it can be shown that **if** $\lim_{x\to a} (f \circ g)(x)$ exists then $\lim_{x\to a} (f \circ g)(x) = f(L)$. But we need some condition such as f is continuous at L to deduce that $\lim_{x\to a} (f \circ g)(x)$ exists.